Announcements

1) Candidate talk today

3 - 4 CB 2070

Squeeze Theorem: Suppose

$a_n \le b_n \le c_n$ and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L.$$

Then $\lim_{n\to\infty} b_n = L$

proof: Since $(c_n)_{n\in\mathbb{N}} \to L$, $\exists$

$N_1 \in \mathbb{N}$ with

$$|c_n - L| < \varepsilon$$

$\forall \; n \ge N_1$.

Since $(a_n)_{n \in \mathbb{N}} \to L$,

$\exists N_2$ with

$$|a_n - L| < \varepsilon$$

$$\forall n \geq N_2.$$

Consider $|b_n - L| < \varepsilon$

equivalent to

$$-\varepsilon < b_n - L < \varepsilon.$$

since $b_n \leq c_n$

$$b_n - L \leq c_n - L$$
$$\leq |c_n - L|$$

and since $a_n \leq b_n$

$$L - b_n$$
$$\leq L - a_n$$
$$\leq |L - a_n|.$$

Now let $N = \max(N_1, N_2)$

Then $|L - a_n|$ and $|c_n - L|$ are both less than ε $\forall$ $n \geq N$.

Then we have, $\forall\, n \geq N$,

$b_n - L < \varepsilon$ and

$L - b_n < \varepsilon$.

This implies $|b_n - L| < \varepsilon$.

$\square$

You can now use the squeeze theorem whenever applicable.

Example 1: (calc squeeze)

Show $\lim_{n \to \infty} \frac{\sin(n)}{n} = 0$.

Consider $b_n = \frac{\sin(n)}{n}$

$a_n = \frac{-1}{n}$

$c_n = \frac{1}{n}$.

Since $-1 \leq \sin(n) \leq 1 \ \forall n \in \mathbb{N}$, dividing by n yields.

$$-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$$

$$\begin{array}{ccc} -\frac{1}{n} & \frac{\sin(n)}{n} & \frac{1}{n} \\ \parallel & \parallel & \parallel \\ a_n & b_n & c_n \end{array}$$

We know $\lim\limits_{n\to\infty} c_n = \lim\limits_{n\to\infty} a_n = 0$,

So by the squeeze Theorem,

$$\lim_{n\to\infty} b_n = 0.$$

Subsequences

$\frac{(-1)^n+1}{2} = a_n$ does not

converge, but...

the even terms are all equal to one, and the odd terms are all equal to zero.

So roughly, "half" of the sequence converges to zero, and the other half converges to one.

Definition: (subsequence)

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence.

Let $n_1 < n_2 < n_3 < n_4 < \cdots$

be an increasing sequence

of natural numbers $(n_k)_{k\in\mathbb{N}}$.

A subsequence of $(a_n)_{n\in\mathbb{N}}$

is a new sequence

$\boxed{(a_{n_k})_{k\in\mathbb{N}}}$ comprised

of terms of the original sequence

whose index is n_k for some $k\in\mathbb{N}$.

Example 2: Let $a_n = \frac{(-1)^n + 1}{2}$.

Consider the subsequence generated by $n_1 = 2$, $n_2 = 4$, $n_3 = 6$,

$n_k = 2k$ for $k \in \mathbb{N}$.

Then $(a_{n_k})_{k\in\mathbb{N}} = (a_{2k})_{k\in\mathbb{N}}$

and $a_{2k} = \frac{(-1)^{2k}+1}{2} = 1 \quad \forall\, k \in \mathbb{N}$.

With the same original sequence, let

$n_1 = 1$, $n_2 = 3$, $n_3 = 5$

$n_k = 2k-1$ for $k \in \mathbb{N}$,

Then $a_{n_k} = a_{2k-1} = \dfrac{(-1)^{2k-1} + 1}{2}$

$= 0$

$\forall\, k \in \mathbb{N}$.

Same sequence, now let n_k = the k^{th} prime number.

$n_1 = 2$, $n_2 = 3$, $n_3 = 5$

$$a_{n_k} = \begin{cases} 1 & k = 2 \\ 0 & k > 2 \end{cases}$$

because 2 is the only even prime!

In general when you see a problem on subsequences, you can usually get away with evens and odds in this class

Proposition: (Subsequences of convergent sequence)

Let $(a_n)_{n\in\mathbb{N}}$ converge to L. Then any subsequence also converges to L.

Proof: Let $\varepsilon > 0$. Then since $\lim_{n\to\infty} a_n = L$, $\exists\ N \in \mathbb{N}$ so that $|a_n - L| < \varepsilon\ \forall\ n \geq N$.

Let $n_1 < n_2 < n_3 < \cdots$ determine a subsequence $(a_{n_k})_{k \in \mathbb{N}}$ of $(a_n)_{n \in \mathbb{N}}$.

Then since the indices of the subsequence are increasing, $\exists\, K \in \mathbb{N},\ n_K > N$.

Then for all $k \geq K$, $n_k \geq n_K$, so $n_k \geq N$, hence

$$|a_{n_k} - L| < \varepsilon \qquad \square$$

Theorem: (Bolzano-Weierstrass)

Any bounded sequence admits a convergent subsequence.

proof: next class!

Proving Some Properties of Sequences

Assume $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} b_n = M$.

Show

1) $\lim_{n\to\infty} (a_n + b_n) = L + M$

proof Observe that

$$|a_n + b_n - (L+M)|$$

$$= |(a_n - L) + (b_n - M)|$$

$$\leq |a_n - L| + |b_n - M|$$

Let $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = L$,

$\exists$ $N_1 \in \mathbb{N}$ with

$$\boxed{|a_n - L| < \frac{\varepsilon}{2}} \quad \forall\, n \geq N_1$$

Since $\lim_{n\to\infty} b_n = M$, $\exists$ $N_2 \in \mathbb{N}$

with $\boxed{|b_n - M| < \frac{\varepsilon}{2}} \quad \forall\, n \geq N_2$

Let $N = \max\{N_1, N_2\}$. Then

from the previous page,

$$|(a_n+b_n)-(L+M)|$$

$$\leq |a_n - L| + |b_n - M|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall n \geq N \quad \square$$